MATH 218: Elementary Linear Algebra with Applications

Spring 2015-2016, Quiz 2, Duration: 60 min.

Name: Solution

Exercise	Points	Scores
1	17	
2	20	
3	15	
4	15	
5	23	
6	10	
Total	100	

INSTRUCTIONS:

(a) Explain your answers in detail and clearly to ensure full credit.

(b) No book. No notes. No calculator.

Exercise 1. Prove or disprove using an explicit counterexample that the following sets are subspaces (either the set is a subspace and you have to prove it or the set is not a subspace and you have to provide a counterexample):

(a) **(5 points)** $U_1 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 0 \right\}$.

Note that $U_1 = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$ since $x^2+y^2+z^2=0$ if and only if $x=y=z=0$.
We know from class that $U_1 = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$ is a subspace.

(b) **(6 points)** $U_2 = \{P \in \mathbb{R}_1[X] \mid P'(0) \geq 0\}$.

No since $X \in U_2$ but $-X \notin U_2$.

(c) **(6 points)** $U_3 = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2\times 2}(\mathbb{R}) \mid ad - c = 0 \right\}$.

No since $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in U_3$ but
$2\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \notin U_3$ $(2 \cdot 2 - 2 \neq 0)$

Exercise 2. Let A be a square $n \times n$ matrix.

(1) **(5 points)** Recall the definition of the kernel $\mathrm{Ker}A$ of A.

$$\mathrm{Ker}A = \{ X \in \mathbb{R}^n \mid AX = 0 \}, \quad 0 = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

(2) **(3 points)** Prove that if $X \in \mathrm{Ker}A$ then $X \in \mathrm{Ker}A^2$.

Check $A^2 X = AAX = A(\underbrace{AX}_{=0 \text{ since } X \in \mathrm{Ker}})$

$= A0$

matrix property $\Rightarrow = 0$

(3) **(3 points)** Find a matrix A such that $\dim \mathrm{Ker}A = \dim \mathrm{Ker}A^2$.

$$A = I = \begin{pmatrix} 1 & & (0) \\ & \ddots & \\ (0) & & 1 \end{pmatrix} \text{ since } A^2 = A.$$

(4) **(3 points)** Find a matrix A such that $\dim \mathrm{Ker}A < \dim \mathrm{Ker}A^2$.

$$A = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & 0 & 0 \end{pmatrix}$$ dim Ker $A = n-1$ ($n-1$ free variables).

$$A^2 = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$$ dim Ker $A = n$ (n free variables)

(5) **(2 points)** Find a matrix A such that $\mathrm{Ker}A \neq \mathrm{Ker}A^2$.

It was the point of the question

Same A than (4), since $\dim \ker A < \dim \ker A^2$ we have $\ker A \neq \ker A^2$ (otherwise dim would be the same).

(6) **(4 points)** Is there a matrix A such that $0 = \dim \mathrm{Ker}A < \dim \mathrm{Ker}A^2 = 1$?

No. ~~If~~

$0 = \dim \ker A \iff A$ invertible

$\iff A^2$ invertible

$\iff \dim \ker A^2 = 0$

can't be 1 if $\dim \ker A = 0$

Exercise 3. (15 points) Consider the following matrix

$$A = \begin{pmatrix} 1 & -1 & 0 & 2 & 1 \\ 0 & 1 & -1 & 0 & -1 \\ -1 & 0 & 1 & 0 & 2 \end{pmatrix}.$$

Determine a basis of $\operatorname{Ker} A$.

$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \in \operatorname{Ker} A$ ie $AX = \vec{0}$

Solve using Gauss Method:

$$\left(\begin{array}{ccccc|c} 1 & -1 & 0 & 2 & 1 & 0 \\ 0 & 1 & -1 & 0 & -1 & 0 \\ -1 & 0 & 1 & 0 & 2 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccccc|c} 1 & -1 & 0 & 2 & 1 & 0 \\ 0 & 1 & -1 & 0 & -1 & 0 \\ 0 & -1 & 1 & 2 & 3 & 0 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccccc|c} \boxed{1} & -1 & 0 & 2 & 1 & 0 \\ 0 & \boxed{1} & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & \boxed{2} & 2 & 0 \end{array}\right)$$

x_1, x_2, x_4 leading, $x_3 = s$, $x_5 = t$ free.

$\Rightarrow x_4 = -t$, $x_2 = s + t$ and

$x_1 = x_2 - 2x_4 - x_5 = s + t + 2t - t$

$= s + 2t$

$$\Rightarrow X = \begin{pmatrix} s+2t \\ s+t \\ s \\ -t \\ t \end{pmatrix} = s \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

$\longrightarrow$

Basis of Ker A is $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$

Exercise 4. (15 points) Determine depending on the value of a a basis <u>and</u> the dimension of

$$\operatorname{span}\left\{\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}, \begin{pmatrix}1 & -1\\ a & 1\end{pmatrix}, \begin{pmatrix}0 & a\\ -1 & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right\}.$$

Use the strategy seen in class:

write

$$\alpha_1\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix} + \alpha_2\begin{pmatrix}1 & -1\\ a & 1\end{pmatrix} + \alpha_3\begin{pmatrix}0 & a\\ -1 & 1\end{pmatrix} + \alpha_4\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$\Rightarrow \begin{aligned} \alpha_1 + \alpha_2 + \alpha_4 &= 0\\ -\alpha_1 - \alpha_2 + a\alpha_3 &= 0\\ a\alpha_2 - \alpha_3 &= 0\\ \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 &= 0 \end{aligned}$$

$$\begin{pmatrix} 1 & 1 & 0 & 1\\ -1 & -1 & a & 0\\ 0 & a & -1 & 0\\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

$$\rightarrow \left(\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 0\\ -1 & -1 & a & 0 & 0\\ 0 & a & -1 & 0 & 0\\ 1 & 1 & 1 & 1 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 0\\ 0 & 0 & a & 1 & 0\\ 0 & a & -1 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 0\\ 0 & a & -1 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & a & 1 & 0 \end{array}\right)$$

Case 1: $a=0$ $\left(\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$

$\to \left(\begin{array}{cccc|c} \boxed{1} & 1 & 0 & 1 & 0 \\ 0 & 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$ $\Rightarrow d_2$ free variable
$\Rightarrow d_1, d_3, d_4$ leading.

$\Longrightarrow$ Basis is $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & a \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
(we have removed the second matrix).
and dim $= 3$

Case 2: $a \neq 0$ $\left(\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 0 \\ 0 & a & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & a & 1 & 0 \end{array}\right)$

$\to \left(\begin{array}{cccc|c} \boxed{1} & 1 & 0 & 1 & 0 \\ 0 & \boxed{a} & -1 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 \end{array}\right)$ No free.

$\to$ Basis is $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ a & 1 \end{pmatrix}, \begin{pmatrix} 0 & a \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

dim $= 4$.

Exercise 5. Consider the set $U = \{P \in \mathbb{R}_3[X] \mid P(0) = P'(0)\}$.

(a) **(7 points)** Prove that U is a subspace of $\mathbb{R}_3[X]$.

i. $0 \in U$ since $0(0) = 0'(0)$.

ii. let $\alpha \in \mathbb{R}$, let $P, Q \in U$ (i.e $P(0) = P'(0)$, $Q(0) = Q'(0)$)

Compute:

$$(\alpha P + Q)(0) = \alpha P(0) + Q(0)$$
$$= \alpha P'(0) + Q'(0)$$
$$= (\alpha P + Q)'(0)$$

So $\alpha P + Q \in U$.

i & ii $\Rightarrow$ U subspace of $\mathbb{R}_3[X]$.

(b) **(3 points)** Let $a_0 + a_1X + a_2X^2 + a_3X^3 \in U$. Show that $a_0 = a_1$.

Note that $P(0) = a_0$

$P'(0) = a_1$

So if $P = a_0 + a_1 X + a_2 X^2 + a_3 X^3 \in U$

we have $a_0 = a_1$.

Note $1+X \in U$
$X^2 \in U$
$X^3 \in U$

(c) **(7 points)** Show that $\{1+X, X^2, X^3\}$ is a basis of U.

i. $1+X, X^2, X^3$ ll:

write $\alpha_1(1+X)+\alpha_2 X^2+\alpha_3 X^3=0$

$\alpha_1+\alpha_1 X+\alpha_2 X^2+\alpha_3 X^3=0$

$\Rightarrow \alpha_1=\alpha_2=\alpha_3=0.$

$\Rightarrow 1+X, X^2, X^3$ ll.

ii. $\text{span}\{1+X, X^2, X^3\}=U$

Let $P \in U$. by (b) P is written

$P=a_0+a_0 X+a_2 X^2+a_3 X^3$

$=a_0(1+X)+a_2 X^2+a_3 X^3$

$\Rightarrow P \in \text{span}\{1+X, X^2, X^3\}$.

$\Rightarrow \text{span}\{1+X, X^2, X^3\}=U.$

i & ii $\Rightarrow 1+X, X^2, X^3$ is a basis of U.

vector in the basis

(d) **(2 points)** What is the dimension of U?

By (c) $\dim U = 3$

Note that the 3 vectors are in U

(e) **(4 points)** Does $\operatorname{span}\{\underbrace{-X^2+X^3}_{1}, \underbrace{2+2X+12X^3}_{2}, \underbrace{1+X}_{3}\} = U$?

Since $\dim U = 3$ the span of 3 poly is $U \iff$ 3 poly $\perp\!\!\!\perp$.

Write

$$\lambda_1(-X^2+X^3) + \lambda_2(2+2X+12X^3) + \lambda_3(1+X) = $$

$$(2\lambda_2+\lambda_3) + (2\lambda_2+\lambda_3)X + -\lambda_1 X^2 + (\lambda_1+12\lambda_2)X^3 =$$

$$\Rightarrow \quad 2\lambda_2+\lambda_3 = 0 \Rightarrow \lambda_3 = 0$$

$$\lambda_1 = 0$$

$$\lambda_1 + 12\lambda_2 = 0 \Rightarrow \lambda_2 = 0$$

$$\Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = 0$$

$$\Rightarrow -X^2+X^3,\ 2+2X+12X^3,\ 1+X \ \perp\!\!\!\perp$$

$$\Rightarrow \operatorname{span}\{-,-,-\} = U$$

$$A = \begin{pmatrix} c_1 & c_2 & \cdots & c_n \end{pmatrix}$$

1st colum ....

Exercise 6. Let A be an $m \times n$ matrix such that the columns of A are linearly independent.

(a) **(5 points)** Prove that $\mathrm{Ker}A = \{0\}$. Explain your answers in detail and clearly to ensure full credit.

Let $X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathrm{Ker}\,A$.

So $AX = \begin{pmatrix} c_1 & \cdots & c_n \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$ of $\mathbb{R}^m$

$\Longrightarrow \quad x_1 c_1 + \cdots + x_n c_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$

Since $c_1, \ldots, c_n$ LI we have $x_1 = x_2 = \cdots = x_n = 0$

So $X = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$ of $\mathbb{R}^n$

Ccl: The only element of $\mathrm{Ker}\,A$ is $\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$ of $\mathbb{R}^n$

(b) **(5 points)** Find dim $\mathrm{Im}A$. Explain your answers in detail and clearly to ensure full credit.

method 1: $\mathrm{Im}\,A = \mathrm{span}\{c_1, \ldots, c_n\}$

Since $c_1, \ldots, c_n$ LI the $c_1, \ldots, c_n$ form a basis of $\mathrm{Im}\,A$

$\Rightarrow \dim \mathrm{Im}\,A = n$.

method 2: formula seen in class

variables = # free + # leading

$n = \dim \underbrace{\mathrm{Ker}\,A}_{0} + \dim \mathrm{Im}\,A$

$\Rightarrow \dim \mathrm{Im}\,A = \cancel{0}\, n$